

On a random recursion related to absorption times of death Markov chains

Alex Iksanov*

*Faculty of Cybernetics, National T. Shevchenko University,
01033 Kiev, Ukraine*

Martin Möhle†

*Mathematical Institute, University of Düsseldorf,
40225 Düsseldorf, Germany*

October 30, 2007

Abstract

Let X_1, X_2, \dots be a sequence of random variables satisfying the distributional recursion $X_1 = 0$ and $X_n \stackrel{d}{=} X_{n-I_n} + 1$ for $n = 2, 3, \dots$, where I_n is a random variable with values in $\{1, \dots, n-1\}$ which is independent of X_2, \dots, X_{n-1} . The random variable X_n can be interpreted as the absorption time of a suitable death Markov chain with state space $\mathbb{N} := \{1, 2, \dots\}$ and absorbing state 1, conditioned that the chain starts in the initial state n .

This paper focuses on the asymptotics of X_n as n tends to infinity under the particular but important assumption that the distribution of I_n satisfies $\mathbb{P}\{I_n = k\} = p_k / (p_1 + \dots + p_{n-1})$ for some given probability distribution $p_k = \mathbb{P}\{\xi = k\}$, $k \in \mathbb{N}$.

Depending on the tail behaviour of the distribution of ξ , several scalings for X_n and corresponding limiting distributions come into play, among them stable distributions and distributions of exponential integrals of subordinators.

The methods used in this paper are mainly probabilistic. The key tool is a coupling technique which relates the distribution of X_n to a random walk, which explains, for example, the appearance of the Mittag-Leffler distribution in this context.

*e-mail address: iksan@unicyb.kiev.ua

†e-mail address: moehle@math.uni-duesseldorf.de

The results are applied to describe the asymptotics of the number of collisions for certain beta-coalescent processes.

Keywords: absorption time; beta coalescent; coupling; exponential integrals; Mittag-Leffler distribution; random recursive equation; stable limit; subordinator

AMS 2000 Mathematics Subject Classification: Primary 60F05; 60G50 Secondary 05C05; 60E07

1 Introduction and main results

Consider a death Markov chain $\{Z_k : k \in \mathbb{N}_0 := \{0, 1, \dots\}\}$ with state space $\mathbb{N} := \{1, 2, \dots\}$ and transition probabilities $\pi_{ij} > 0$ for $i, j \in \mathbb{N}$ with $j < i$ and $\pi_{ij} = 0$ otherwise. For $n \in \mathbb{N}$, define

$$X_n := \inf\{k \geq 1 : Z_k = 1 \text{ given } Z_0 = n\}.$$

Note that $X_n \in \{1, 2, \dots, n-1\}$ almost surely.

Surprisingly, there seems to be very little known about the asymptotic behavior of X_n as n tends to infinity. To our knowledge, [8] is one paper addressing this question. However, the assumptions and the approach to be presented here are completely different from those in [8].

The random variable X_n can be interpreted as the number of parts of the random composition C_{n-1} of the integer $n-1$, where the parts of the composition are (by definition) the decrements of the Markov chain $\{Z_k : k \in \mathbb{N}_0\}$. There are several important articles in the literature ([3, 14, 15, 16]) with asymptotic results on random compositions. However, in all these papers the consistency of the random compositions for different values of n is a crucial assumption, i.e. all these papers focus on so called random composition structures. We do not assume this consistency property here. Hence, our setting differs significantly from that in the mentioned papers.

The key observation is that X_n satisfies the distributional recursion $X_1 = 0$ and

$$X_n \stackrel{d}{=} X_{n-I_n} + 1, \quad n \in \{2, 3, \dots\}, \quad (1)$$

where I_n is a random variable independent of X_2, \dots, X_{n-1} with distribution $\mathbb{P}\{I_n = k\} = \pi_{n,n-k}$, $k \in \{1, \dots, n-1\}$. The crucial assumption for the paper is that

$$\mathbb{P}\{I_n = k\} = \frac{p_k}{p_1 + \dots + p_{n-1}}, \quad k, n \in \mathbb{N}, k < n, \quad (2)$$

for some proper and non-degenerate probability distribution

$$p_k := \mathbb{P}\{\xi = k\}, \quad k \in \mathbb{N}, \quad p_1 > 0. \quad (3)$$

Throughout the paper $r(\cdot) \sim s(\cdot)$ means that $r(\cdot)/s(\cdot) \rightarrow 1$ as the argument tends to infinity. The symbols \xrightarrow{d} , \Rightarrow , and \xrightarrow{P} denote convergence in law, weak convergence, and convergence in probability, respectively, and $X_n \xrightarrow{d} (\Rightarrow, \xrightarrow{P})X$ means that the limiting relation holds when $n \rightarrow \infty$. With L we always denote a function slowly varying at infinity.

Our main results given next are concerned with the limiting behaviour of X_n as $n \rightarrow \infty$. We begin with a weak law of large numbers.

Theorem 1.1. *If $\sum_{m=1}^n \sum_{k=m}^{\infty} p_k \sim L(n)$ for some function L slowly varying at ∞ , then, as $n \rightarrow \infty$,*

$$\frac{X_n}{\mathbb{E}X_n} \xrightarrow{P} 1 \quad (4)$$

and $\mathbb{E}X_n \sim n/L(n)$. In particular, if

$$m := \mathbb{E}\xi < \infty, \quad (5)$$

then $\mathbb{E}X_n \sim n/m$. If (5) holds, and if there exists a sequence of positive numbers $\{a_n : n \in \mathbb{N}\}$ such that $X_n/a_n \xrightarrow{P} 1$ as $n \rightarrow \infty$, then $a_n \sim n/m$.

To formulate further results we need some more notation. For $C > 0$ and $\alpha \in [1, 2]$ let μ_α be an α -stable distribution with characteristic function $\psi_\alpha(t)$, $t \in \mathbb{R}$ of the form

$$\begin{aligned} \exp\{-|t|^\alpha C \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sgn}(t))\}, \quad 1 < \alpha < 2; \\ \exp\{-|t|C(\pi/2 - i \log |t| \operatorname{sgn}(t))\}, \quad \alpha = 1; \\ \exp(-(C/2)t^2), \quad \alpha = 2. \end{aligned}$$

In the case when (5) holds, Theorem 1.2 provides necessary and sufficient conditions ensuring that X_n , properly normalized and centered, possesses a weak limit.

Theorem 1.2. *If $m := \mathbb{E}\xi < \infty$, then the following assertions are equivalent.*

- (i) *There exist sequences of numbers $\{a_n, b_n : n \in \mathbb{N}\}$ with $a_n > 0$ and $b_n \in \mathbb{R}$ such that, as $n \rightarrow \infty$, $(X_n - b_n)/a_n$ converges weakly to a non-degenerate and proper probability law.*

(ii) Either $\sigma^2 := \mathbb{D}\xi < \infty$, or $\sigma^2 = \infty$ and for some $\alpha \in [1, 2]$ and some function L slowly varying at ∞ ,

$$\sum_{k=1}^n k^2 p_k \sim n^{2-\alpha} L(n), \quad n \rightarrow \infty. \quad (6)$$

If $\sigma^2 < \infty$, then, with $b_n := n/m$ and $a_n := (m^{-3}C^{-1}\sigma^2 n)^{1/2}$, the limiting law is μ_2 (normal with mean zero and variance C).

If $\sigma^2 = \infty$ and (6) holds with $\alpha = 2$, then, with $b_n := n/m$ and $a_n := m^{-3/2}c_n$, where c_n is any sequence satisfying $\lim_{n \rightarrow \infty} nL(c_n)/c_n^2 = C$, the limiting law is μ_2 .

If $\sigma^2 = \infty$ and (6) holds with $\alpha \in [1, 2)$, then, with $b_n := n/m$ and $a_n := m^{-(\alpha+1)/\alpha}c_n$, where c_n is any sequence satisfying

$$\lim_{n \rightarrow \infty} \frac{nL(c_n)}{c_n^\alpha} = \frac{\alpha}{2-\alpha}C,$$

the limiting law is μ_α .

Remark 1.3. For $\sigma^2 < \infty$, the same weak convergence result for X_n was obtained in Theorem 4.1 in [8] in a setting more general than ours. Note that for $\alpha \in [1, 2)$, (6) is equivalent to $\mathbb{P}\{\xi \geq n\} \sim (2-\alpha)n^{-\alpha}L(\alpha)/\alpha$, $n \rightarrow \infty$.

If the mean of ξ is infinite, the following Theorem 1.4 (Theorem 1.5) points out conditions ensuring that X_n , properly normalized (and centered), possesses a weak limit.

Theorem 1.4. *Suppose that for some $\alpha \in (0, 1)$ and some function L slowly varying at ∞*

$$\mathbb{P}\{\xi \geq n\} = \sum_{k=n}^{\infty} p_k \sim \frac{L(n)}{n^\alpha}, \quad n \rightarrow \infty. \quad (7)$$

Then, as $n \rightarrow \infty$,

$$\frac{L(n)}{n^\alpha} X_n \xrightarrow{d} \int_0^\infty e^{-U_t} dt, \quad (8)$$

where $\{U_t : t \geq 0\}$ is a subordinator with zero drift and Lévy measure

$$\nu(dt) = \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}} dt, \quad t > 0. \quad (9)$$

It is instructive to present two proofs for Theorem 1.4, namely a probabilistic proof and an analytic proof. The probabilistic proof given in Section 5 reveals a relation between Eq. (12) and perpetuities. The analytic proof of Theorem 1.4 presented in Section 6 starts with the distributional recursion (1), which implies that, for fixed $k \in \mathbb{N}$, the sequence $\{\mathbb{E}X_n^k : n \in \mathbb{N}\}$ satisfies another recursion. The structure of this last recursion permits a relatively simple asymptotic analysis of $\mathbb{E}X_n^k$. In this way it is possible to derive the convergence of the moments

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\frac{L(n)}{n^\alpha} X_n \right)^k = \mathbb{E} \left(\int_0^\infty e^{-U_t} dt \right)^k, \quad k \in \mathbb{N},$$

which by a standard argument leads to (8).

Theorem 1.5. *Suppose that $\mathbb{E}\xi = \infty$ and that for some function L slowly varying at ∞*

$$\mathbb{P}\{\xi \geq n\} = \sum_{k=n}^\infty p_k \sim \frac{L(n)}{n}. \quad (10)$$

Let c be any positive function satisfying $\lim_{x \rightarrow \infty} xL(c(x))/c(x) = 1$ and set $\psi(x) := x \int_0^{c(x)} \mathbb{P}\{\xi > y\} dy$. Let $b(x)$ be any positive function satisfying

$$b(\psi(x)) \sim \psi(b(x)) \sim x,$$

and set $a(x) := x^{-1}b(x)c(b(x))$. Then, $(X_n - b(n))/a(n)$ converges weakly to the stable distribution μ_1 with $C = 1$.

In the literature there exist two standard approaches to studying distributional recursions. One approach is purely analytic and based on a singularity analysis of generating functions (see, for example, [11, 23]). The other approach, called *contraction method*, is more probabilistic (see [22, 27, 28]). It was remarked in [20] that recursions (1) which satisfy (2) can be successfully investigated by using probabilistic methods alone (completely different from contraction methods). The present work extends ideas laid down in [20] for the particular case

$$\mathbb{P}\{I_n = k\} = \frac{n}{n-1} \frac{1}{k(k+1)}, \quad k \in \{1, \dots, n-1\}.$$

The basic steps of the technique exploited can be summarized as follows.

Let ξ_1, ξ_2, \dots be independent copies of a random variable ξ with distribution (3). Define $S_0 := 0$, $S_n := \xi_1 + \dots + \xi_n$ and $N_n := \inf\{k \geq 1 : S_k \geq n\}$,

$n \in \mathbb{N}$. Since $I_n \xrightarrow{d} \xi$, one may expect that the limiting behaviour of X_n and N_n is similar, or at least that the limiting behaviour of the latter will influence that of the former. To make this intuition precise, on the probability space where S_k and N_n are defined, we will construct (Section 2) random variables M_n with the same distributions as X_n . Similarity in the limiting behaviour of M_n and N_n is well indicated by asymptotic properties of their difference. In particular, we will prove the following.

(a) If $\mathbb{E}\xi < \infty$, then $M_n - N_n$ weakly converges. Therefore, M_n , properly normalized and centered, possesses a weak limit if and only if the same is true for N_n .

(b) Assume now that $\mathbb{E}\xi = \infty$. (b1) If $\sum_{k=n}^{\infty} p_k \sim L(n)/n$ and if $(N_n - b_n)/a_n$ weakly converges to some μ , then $(M_n - N_n)/a_n \xrightarrow{P} 0$ which proves that $(M_n - b_n)/a_n$ weakly converges to μ . Thus in cases (a) and (b1) a weak behaviour of M_n and N_n is the same. (b2) If, for some $\alpha \in (0, 1)$, $\sum_{k=n}^{\infty} p_k \sim n^{-\alpha} L(n)$ and N_n/a_n weakly converges to some ν_1 , then $(M_n - N_n)/a_n$ weakly converges to some ν_2 . Even though, the argument exploited above does not apply, it will be proved that M_n/a_n weakly converges to $\nu_3 \neq \nu_1$. Thus in this latter case a weak behaviour of M_n is not completely determined by that of N_n . Now it is influenced by the weak behaviour of both N_n and $n - S_{N_n-1}$ to, approximately, the same extent. This observation can be explained as follows. The probability of one big jump of S_n in comparison to cases (a) and (b1) is higher, and therefore the epoch N_n comes more "quickly". As a consequence, a contribution to M_n of the number of jumps in the sequence $R_k^{(n)}$ (defined in Section 2), while $R_k^{(n)}$ is travelling from $R_{N_n-1}^{(n)} = S_{N_n-1}$ to $n - 1$, gets significant.

It remains to review structural units of the paper not mentioned so far. In Section 3 we investigate both the univariate and the bivariate weak behaviour of $(N_n, n - S_{N_n-1})$, and discuss their relation to exponential integrals of subordinators. Theorem 1.2, 1.1 and 1.5 are proved in Section 4, 7 and 8 respectively. In Section 9 our main results apply to the number of collisions in certain beta coalescent processes. Possible generalizations of the results obtained and some directions for future work are discussed in the final Section 10.

2 A coupling

Fix $n \in \mathbb{N}$. Define $R_0^{(n)} := 0$ and

$$R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \quad k \in \mathbb{N}.$$

Note that the sequence $\{R_k^{(n)} : k \in \mathbb{N}_0\}$ is non-decreasing. Let

$$M_n := \#\{i \in \mathbb{N} : R_{i-1}^{(n)} \neq R_i^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}}$$

denote the number of jumps of the process $\{R_k^{(n)} : k \in \mathbb{N}_0\}$. Note that $M_1 = 0$ and that $1 \leq M_n \leq n-1$ for $n \geq 2$. As $p_1 > 0$, it follows from Lemma 1 in [20] that the distribution of M_n satisfies the same recursion (1) as X_n . Hence, the following lemma holds.

Lemma 2.1. *For each $n \in \mathbb{N}$, the distribution of M_n coincides with the distribution of the random variable X_n introduced in Section 1.*

Fix $m, i \in \mathbb{N}$. Define $\widehat{R}_0^{(m)}(i) := 0$,

$$\widehat{R}_k^{(m)}(i) := \widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} 1_{\{\widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} < m\}}, \quad k \in \mathbb{N},$$

and

$$\widehat{M}_n(i) := \sum_{l=0}^{\infty} 1_{\{\widehat{R}_l^{(n)}(i) + \xi_{i+l+1} < n\}}, \quad n \in \mathbb{N}_0.$$

Our probabilistic proof of Theorem 1.4 relies upon the following decomposition (12).

Lemma 2.2. *For fixed $n \in \mathbb{N}$ and any $i \in \mathbb{N}$,*

$$\widehat{M}_n(i) \stackrel{d}{=} M_n, \tag{11}$$

and

$$M_n - N_n + 1 = \widehat{M}_{n-S_{N_n-1}}(N_n) \stackrel{d}{=} M'_{n-S_{N_n-1}}, \tag{12}$$

where $\{M'_n : n \in \mathbb{N}\}$ has the same law as $\{M_n : n \in \mathbb{N}\}$ and is independent of $(N_n, n - S_{N_n-1})$.

Proof. We have

$$\begin{aligned} M_n &= \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}} = \sum_{l=0}^{N_n-2} 1 + \sum_{l=N_n}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}} \\ &= N_n - 1 + \sum_{l=0}^{\infty} 1_{\{\widehat{R}_l^{(n-S_{N_n-1})}(N_n) + \xi_{N_n+l+1} < n-S_{N_n-1}\}} \\ &= N_n - 1 + \widehat{M}_{n-S_{N_n-1}}(N_n), \end{aligned}$$

and the first equality in (12) follows. For any fixed $m \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{P}\{\widehat{M}_{n-S_{N_n-1}}(N_n) = m\} \\
&= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\{\widehat{M}_{n-j}(i) = m, N_n = i, S_{N_n-1} = j\} \\
&= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\left\{\sum_{l=0}^{\infty} 1_{\{\widehat{R}_l^{(n-j)}(i) + \xi_{i+l+1} < n-j\}} = m, N_n = i, S_{N_n-1} = j\right\}.
\end{aligned}$$

The sequence $\{\widehat{R}_l^{(n-j)}(i) + \xi_{i+l+1} : l \in \mathbb{N}_0\}$ is independent of $1_{\{N_n=i, S_{N_n-1}=j\}}$ and has the same law as $\{(R_l^{(n-j)})' + \xi'_{l+1} : l \in \mathbb{N}_0\}$, where $\{(R_l^{(\cdot)})' : l \in \mathbb{N}_0\}$ is constructed in the same way as the sequence without "prime" by using $\{\xi'_k : k \in \mathbb{N}\}$, an independent copy of $\{\xi_k : k \in \mathbb{N}\}$. This implies (11) and

$$\begin{aligned}
& \mathbb{P}\{\widehat{M}_{n-S_{N_n-1}}(N_n) = m\} \\
&= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\left\{\sum_{l=0}^{\infty} 1_{\{(R_l^{(n-j)})' + \xi'_{l+1} < n-j\}} = m\right\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\
&= \mathbb{P}\left\{\sum_{l=0}^{\infty} 1_{\{(R_l^{(n-S_{N_n-1})})' + \xi'_{l+1} < n-S_{N_n-1}\}} = m\right\} = \mathbb{P}\{M'_{n-S_{N_n-1}} = m\},
\end{aligned}$$

and the second equality in distribution in (12) follows. \square

3 Results on N_n and $n - S_{N_n-1}$: case $m = \infty$

3.1 Univariate results

Below necessary and sufficient conditions are collected ensuring that a properly normalized (without centering) N_n weakly converges to a non-degenerate law (Proposition 3.1) and to δ_1 (Proposition 3.3).

We say that a random variable ξ_α has a Mittag-Leffler distribution θ_α with parameter $\alpha \in [0, 1)$, if

$$\mathbb{E}\xi_\alpha^n = \frac{n!}{\Gamma^n(1-\alpha)\Gamma(1+n\alpha)}, \quad n \in \mathbb{N}.$$

Note that the moments $\mathbb{E}\xi_\alpha^n$, $n \in \mathbb{N}$, uniquely determine the distribution. We also write θ_1 for δ_1 .

Proposition 3.1. *If (7) holds for some $\alpha \in [0, 1)$, then*

$$\frac{n^\alpha}{L(n)} N_n \Rightarrow \theta_\alpha. \quad (13)$$

Conversely, assume that there exist positive real numbers $a(n)$, $n \in \mathbb{N}$, such that $N_n/a(n)$ weakly converges to a non-degenerate law θ . Then $a(n) \sim D (\sum_{k=n}^\infty p_k)^{-1} \sim D n^\alpha / L(n)$ for some constants $D > 0$, $\alpha \in [0, 1)$ and some function L slowly varying at ∞ , and (13) holds.

Remark 3.2. Proposition 3.1 for $\alpha = 0$ demonstrates that Theorem 6 in [13] is wrong. For $\alpha \in (0, 1)$, the implication (7) \Rightarrow (13) is well known (see, for example, Theorem 7 in [13]). Our proof of Proposition 3.1 seems to be new. It uses a technique introduced in [9] and simplified in [6], Theorems 8.11.2 and 8.11.3. Note that N_n is not the occupation time in the sense of Darling and Kac. Thus, before exploiting their approach, we had to prove Lemma 3.4, which is crucial for their technique to work.

Proposition 3.3. *The following conditions are equivalent.*

- (a) $\sum_{m=1}^n \sum_{k=m}^\infty p_k \sim L(n)$ for some L slowly varying at ∞ .
- (b) $1 - \sum_{n=1}^\infty e^{-sn} p_n \sim sL(1/s)$ as $s \downarrow 0$ for some L slowly varying at ∞ .
- (c) *The sequence $\{N_n : n \in \mathbb{N}\}$ is relatively stable, i.e. there exist positive real numbers $a(n)$, $n \in \mathbb{N}$, such that $N_n/a(n) \xrightarrow{P} 1$.*

Moreover, if (a) holds, then $a(n) \sim \mathbb{E}N_n \sim n/L(n)$.

Put $P(s) := \sum_{n=1}^\infty e^{-sn} p_n$, $s \geq 0$, and $h(s) := (1 - P(s))^{-1}$, $s > 0$. For $t \geq 0$ define $N_t := \inf\{k \geq 1 : S_k \geq t\}$. Then $N_t = N_1$ for $t \in [0, 1]$, and $N_t = N_n$ for $t \in (n-1, n]$, $n = 2, 3, \dots$

Lemma 3.4. *Fix $k \in \mathbb{N}$. Then, as $s \downarrow 0$,*

$$s \int_0^\infty e^{-st} \mathbb{E}N_t^k dt \sim k! h^k(s). \quad (14)$$

Proof. For $k \in \{2, 3, \dots\}$ let D_k denote the affine function of $k-2$ positive variables of the form

$$D_k(x_1, x_2, \dots, x_{k-2}) = \gamma_{0,k} + \sum_{i=1}^{k-2} \gamma_{i,k} x_i,$$

with coefficients $\gamma_{i,k} \in \mathbb{R}$, $i \in \{0, 1, \dots, k-2\}$. (These coefficients can be derived explicitly, but their exact values are of no use here.) For convenience, define $b_k(n) := \mathbb{E}N_n^k$, $k \in \mathbb{N}$. We prove by induction on k that

$$b_k(n) = c_k(n) + \sum_{i=1}^{n-1} b_k(n-i) p_i, \quad k \in \mathbb{N}, \quad (15)$$

with $c_1(n) := 1$ and

$$c_k(n) := D_k(b_1(n), \dots, b_{k-2}(n)) + k b_{k-1}(n), \quad k \geq 2.$$

For $k = 1$, Eq. (15) immediately follows from

$$N_n \stackrel{d}{=} 1 + N'_{n-\xi} 1_{\{\xi < n\}}, \quad n = 2, 3, \dots, \quad N_1 = 1, \quad (16)$$

where $\{N'_n : n \in \mathbb{N}\}$ is a copy of $\{N_n : n \in \mathbb{N}\}$ and ξ is independent of N'_2, \dots, N'_{n-1} . Suppose (15) holds for $k \in \{1, 2, \dots, m-1\}$. Then,

$$\begin{aligned} b_m(n) &= \\ &= \sum_{i=0}^{m-2} \binom{m}{i} \mathbb{E}(N'_{n-\xi} 1_{\{\xi < n\}})^i + m \mathbb{E}(N'_{n-\xi} 1_{\{\xi < n\}})^{m-1} + \mathbb{E}(N'_{n-\xi} 1_{\{\xi < n\}})^m \\ &= 1 + m(b_1(n) - 1) + \sum_{i=2}^{m-2} \binom{m}{i} (b_i(n) - D_i(b_1(n), \dots, b_{i-2}(n))) \\ &\quad - m D_{m-1}(b_1(n), \dots, b_{m-3}(n)) + m b_{m-1}(n) + \sum_{i=1}^{n-1} b_m(n-i) p_i. \end{aligned}$$

The first four terms on the right-hand side form an affine function of $b_1(n)$, \dots , $b_{m-2}(n)$, which implies (15) for $k = m$. Therefore, (15) is established.

For $k \in \mathbb{N}$ and $s > 0$ define $B_k(s) := \sum_{n=1}^{\infty} e^{-sn} b_k(n)$ and $C_k(s) := \sum_{n=2}^{\infty} e^{-sn} c_k(n)$. Then, (15) is equivalent to

$$B_k(s) = \frac{e^{-s} + C_k(s)}{1 - P(s)} = h(s)(e^{-s} + C_k(s)), \quad k \in \mathbb{N}, s > 0. \quad (17)$$

We now verify by induction on k that

$$s B_k(s) \sim k! h^k(s), \quad s \downarrow 0, k \in \mathbb{N}. \quad (18)$$

From $C_1(s) = e^{-2s}/(1 - e^{-s})$, $s > 0$, and (17) it follows that

$$s B_1(s) = \frac{s e^{-s} h(s)}{1 - e^{-s}} \sim h(s), \quad s \downarrow 0.$$

Thus, (18) holds for $k = 1$. Suppose (18) holds for $k \in \{1, \dots, m\}$ and check that

$$s B_{m+1}(s) \sim (m+1)! h^{m+1}(s), \quad s \downarrow 0.$$

The induction assumptions imply that

$$s \sum_{i=2}^{\infty} e^{-si} D_{m+1}(b_1(i), \dots, b_{m-1}(i)) = o(h^m(s)), \quad s \downarrow 0.$$

Therefore, by (17),

$$\begin{aligned} s B_{m+1}(s) &= s h(s)(e^{-s} + C_{m+1}(s)) \\ &= s h(s)(e^{-s} + (m+1)(B_m(s) - e^{-s}) \\ &\quad + \sum_{i=2}^{\infty} e^{-si} D_{m+1}(b_1(i), \dots, b_{m-1}(i))) \\ &= h(s)((m+1)s B_m(s) - m s e^{-s} + o(h^m(s))) \\ &\sim (m+1)! h^{m+1}(s), \quad s \downarrow 0, \end{aligned}$$

and (18) is established. It remains to note that

$$\begin{aligned} s \int_0^{\infty} e^{-st} \mathbb{E} N_t^k dt &= s \sum_{j=1}^{\infty} \int_{j-1}^j e^{-st} \mathbb{E} N_j^k dt \\ &= (e^s - 1) \sum_{j=1}^{\infty} e^{-sj} \mathbb{E} N_j^k \sim s B_k(s) \sim k! h^k(s), \quad s \downarrow 0. \end{aligned}$$

□

Proof of Propositions 3.1 and 3.3. For N sufficiently large, define $L(t) := L(n)$ for $t \in (n-1, n]$, $n \in \{N, N+1, \dots\}$. Then, (7) is equivalent to $\mathbb{P}\{\xi > x\} \sim x^{-\alpha} L(x)$, and condition (a) of Proposition 3.3 is equivalent to $\int_0^x \mathbb{P}\{\xi > y\} dy \sim L(x)$. Now, by Corollary 8.1.7 in [6], (7) is equivalent to

$$1 - P(s) \sim \Gamma(1 - \alpha) s^{\alpha} L(1/s), \quad s \downarrow 0,$$

and conditions (a) and (b) of Proposition 3.3 are equivalent. Regarding formally $\Gamma(0)$ as 1, assume that $1 - P(s) \sim \Gamma(1 - \alpha) s^{\alpha} L(1/s)$, $s \downarrow 0$, for some $\alpha \in [0, 1]$ or, equivalently,

$$h(s) \sim \frac{1}{\Gamma(1 - \alpha) s^{\alpha} L(1/s)}, \quad s \downarrow 0.$$

We now proceed exactly as in the proof of Theorem 8.11.2 in [6]. Applying Karamata's theorem ([6], Theorem 1.7.6) to (14) gives

$$\mathbb{E}N_t^k \sim \frac{k!}{\Gamma^k(1-\alpha)\Gamma(1+\alpha k)} \frac{t^{\alpha k}}{L^k(t)}.$$

Therefore,

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{L(t)N_t}{t^\alpha} \right)^k = \frac{k!}{\Gamma^k(1-\alpha)\Gamma(1+\alpha k)}, \quad k \in \mathbb{N}, \quad (19)$$

and, as $t \rightarrow \infty$, $L(t)N_t/t^\alpha \Rightarrow \theta_\alpha$, which implies $L(n)N_n/n^\alpha \Rightarrow \theta_\alpha$.

Assume now that $N_n/a(n) \Rightarrow \theta$, and that either $\theta = \delta_1$, or θ is non-degenerate. As the sequence $\{N_n : n \in \mathbb{N}\}$ is almost surely non-decreasing, $\lim_{n \rightarrow \infty} N_n = \infty$ almost surely and $N_{n+1} \leq N_n + 1$ almost surely, we have $1 \leq N_{n+1}/N_n \leq 1 + 1/N_n$ almost surely. Therefore, $\lim_{n \rightarrow \infty} N_{n+1}/N_n = 1$ almost surely and $\lim_{n \rightarrow \infty} a(n+1)/a(n) = 1$. For $t > 0$ define $a(t) := a(n)$, $t \in (n-1, n]$. Then, by a sandwich argument, $N_t/a(t) \Rightarrow \theta$ as $t \rightarrow \infty$.

If θ is non-degenerate, then from the proof of Theorem 8.11.3 in [6] it follows that $a(t) \sim Dh(1/t)$ for some $D > 0$ and that the function a regularly varies at ∞ with exponent $\alpha \in [0, 1)$. By Corollary 8.1.7,

$$a(n) \sim \frac{D}{\Gamma(1-\alpha) \sum_{k=n}^{\infty} p_k}.$$

Therefore, for some $\alpha \in [0, 1)$, (7) holds. By the direct part of the proposition, (13) holds as well.

If $\theta = \delta_1$, then we use a similar but simpler argument. Let T be an exponentially distributed random variable with mean 1 which is independent of $\{N_t : t \geq 0\}$. As in the proof of Theorem 8.11.3 in [6], each sequence r_n tending to 0 contains a subsequence $\{s_n : n \in \mathbb{N}\}$ satisfying $\lim_{n \rightarrow \infty} s_n = 0$, along which $\lim_{n \rightarrow \infty} a(t/s_n)/h(s_n) = f(t)$ at continuity points of a non-decreasing function f . Therefore, $\lim_{n \rightarrow \infty} a(T/s_n)/h(s_n) = f(T)$ almost surely. From (14) it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\frac{N_{T/s_n}}{h(s_n)} \right)^k = k!, \quad k \in \mathbb{N}. \quad (20)$$

Since $N_{T/s_n}/a(T/s_n) \xrightarrow{P} 1$,

$$\frac{N_{T/s_n}}{h(s_n)} \xrightarrow{P} f(T). \quad (21)$$

Applying Fatou's lemma to (20) with $k = 1$ we conclude that $f(T) < \infty$ almost surely. Also, (20) implies that, for each $k \in \mathbb{N}$, the sequence $\{(N_{T/s_n}/h(s_n))^k : n \in \mathbb{N}\}$ is uniformly integrable which, in conjunction with (21), leads to $\mathbb{E}f^k(T) = k!$, $k \in \mathbb{N}$. Since $\mathbb{E}T^k = k!$, $k \in \mathbb{N}$, and the sequence $\{k! : k \in \mathbb{N}\}$ uniquely determines (exponential) distribution, we conclude that $f(t) = t$, $t > 0$. The same argument as above can be repeated for any sequence like r_n which gives $a(t/s)/h(s) \rightarrow t$ as $s \downarrow 0$ for each fixed $t > 0$. Therefore, $a(t/s)/a(1/s) \rightarrow t$ as $s \downarrow 0$, which means that $a(t) \sim h(1/t) \sim t/L(t)$ as $t \rightarrow \infty$ for some L slowly varying at ∞ . Hence, $1 - P(t) \sim tL(1/t)$ as $t \downarrow 0$. \square

Remark 3.5. Suppose (7) holds for some $\alpha \in [0, 1)$. Then,

$$\lim_{n \rightarrow \infty} \frac{L^k(n)}{n^{\alpha k}} \mathbb{E}N_n^k = \frac{k!}{\Gamma^k(1 - \alpha)\Gamma(1 + \alpha k)}, \quad k \in \mathbb{N}. \quad (22)$$

Suppose condition (a) of Proposition 3.3 holds. Then,

$$\lim_{n \rightarrow \infty} \frac{L^k(n)}{n^k} \mathbb{E}N_n^k = 1, \quad k \in \mathbb{N}. \quad (23)$$

These observations immediately follow from (19). Note also that (22) is a particular case of Corollary 3.3 [26].

The next result is a corollary of Theorem 1.1 and Proposition 3.3.

Corollary 3.6. *Assume that (10) holds. Then, $\mathbb{E}N_n \sim \mathbb{E}M_n \sim n/m(n)$, where $m(x) := \int_0^x \mathbb{P}\{\xi > y\} dy$, $x > 0$. Moreover,*

$$\frac{m(n)N_n}{n} \xrightarrow{P} 1 \quad \text{and} \quad \frac{m(n)M_n}{n} \xrightarrow{P} 1.$$

In particular, $M_n/N_n \xrightarrow{P} 1$.

Proof. Condition (10) ensures that $m(x)$ belongs to the de Haan class II, i.e. $\lim_{x \rightarrow \infty} (m(\lambda x) - m(x))/L(x) = \log \lambda$. In particular, $m(\cdot)$ is slowly varying at ∞ . Since $\sum_{m=1}^n \sum_{k=m}^{\infty} p_k \sim m(n)$, Theorem 1.1 together with Lemma 2.1 imply the result for M_n , and Proposition 3.3 implies the result for N_n . \square

The next result is the key ingredient for our proof of Theorem 1.5. Define $Y_n := n - S_{N_n-1}$, $n \in \mathbb{N}$.

Proposition 3.7. *Assume that (10) holds. Then, for fixed $\delta > 0$,*

$$\mathbb{E}Y_n^\delta \sim \frac{n^\delta L(n)}{\delta m(n)}, \quad (24)$$

where $m(x) := \int_0^x \mathbb{P}\{\xi > y\} dy$, $x > 0$. Furthermore, for functions a and b as used in Theorem 1.5,

$$\frac{b(n)Y_n}{n a(n)} \xrightarrow{P} 0. \quad (25)$$

Proof. In the same way as in the proof of Proposition 3.9 it follows that

$$\mathbb{E}Y_n^\delta = \sum_{k=0}^{n-1} (n-k)^\delta \mathbb{P}\{\xi \geq n-k\} u_k, \quad n \in \mathbb{N},$$

where $u_k := \sum_{i=0}^k \mathbb{P}\{S_i = k\}$, $k \in \mathbb{N}_0$. By Corollary 3.6, $\mathbb{E}N_n \sim n/m(n)$. On the other hand, $\mathbb{E}N_n \sim \sum_{k=0}^n u_k$, $n \in \mathbb{N}$. Thus, $\sum_{k=0}^n u_k \sim n/m(n)$ and, by Corollary 1.7.3 in [6],

$$U(s) := \sum_{n=0}^{\infty} s^n u_n \sim \frac{1}{m((1-s)^{-1})(1-s)} \quad \text{as } s \uparrow 1.$$

By the same Corollary

$$V(s) := \sum_{n=1}^{\infty} s^n n^\delta \mathbb{P}\{\xi \geq n\} \sim \frac{\Gamma(\delta) L((1-s)^{-1})}{(1-s)^\delta} \quad \text{as } s \uparrow 1.$$

Therefore,

$$\sum_{n=1}^{\infty} s^n \mathbb{E}Y_n^\delta = U(s)V(s) \sim \frac{\Gamma(\delta)}{(1-s)^{\delta+1}} \frac{L((1-s)^{-1})}{m((1-s)^{-1})} \quad \text{as } s \uparrow 1.$$

The sequence $\{Y_n : n \in \mathbb{N}\}$ is almost surely non-decreasing which implies that the sequence $\{\mathbb{E}Y_n^\delta : n \in \mathbb{N}\}$ is non-decreasing. Therefore, Corollary 1.7.3 in [6] applies and proves (24). Recall that $\psi(x) = xm(c(x))$ and $c(x) \sim xL(c(x))$. Since $m(x)/L(x) \rightarrow \infty$, $c(x) \rightarrow \infty$ and

$$\frac{\psi(x)}{c(x)} = \frac{xm(c(x))}{c(x)} \sim \frac{m(c(x))}{L(c(x))}$$

as $x \rightarrow \infty$, we conclude that $\psi(x)/c(x) \rightarrow \infty$ as $x \rightarrow \infty$. Therefore,

$$J_1(n) := \frac{b([\psi(n)])}{a([\psi(n)])} = \frac{[\psi(n)]}{c(b([\psi(n)]))} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

where $[x]$ denotes the integer part of x , and

$$J_2(n) := \frac{L([\psi(n)])}{m([\psi(n)])} \frac{b([\psi(n)])}{a([\psi(n)])} \sim \frac{L([nm(c(n))])}{m([nm(c(n))])} \frac{[nm(c(n))]}{nL(c(n))}$$

remains bounded for large n .

Put $v(x) := xa(x)/b(x) = c(b(x))$. For fixed $\delta \in (0, 1)$ and any $\epsilon > 0$ we have, by Markov's inequality and by (24),

$$\mathbb{P}\{Y_{[\psi(n)]} > v([\psi(n)])\epsilon\} \leq \frac{\mathbb{E}Y_{[\psi(n)]}^\delta}{v^\delta([\psi(n)])\epsilon^\delta} \sim \frac{J_2(n)J_1^{\delta-1}(n)}{\delta\epsilon^\delta} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The function v is regularly varying at infinity with exponent 1. Therefore, $\lim_{n \rightarrow \infty} v([\psi(n-1)])/v(\psi(n)) = 1$. Without loss of generality we can assume that v is non-decreasing. If, for large n , $k \in ([\psi(n-1)], [\psi(n)])$, then

$$\frac{Y_k}{v(k)} \leq \frac{Y_{[\psi(n)]}}{v([\psi(n-1)])} \text{ almost surely,}$$

and, by what we have already proved, as $n \rightarrow \infty$, the right-hand side tends to 0 in probability, which proves (25). \square

3.2 Some results on exponential integrals of subordinators

Let $\{Z_t : t \geq 0\}$ be a subordinator with zero drift which is independent of T , an exponentially distributed random variable with mean one. Set $Q := \int_0^T e^{-Z_t} dt$, $M := e^{-Z_T}$, and $A := \int_T^\infty e^{-Z_t} dt$. First of all, note that

$$\begin{aligned} A_\infty &:= \int_0^\infty e^{-Z_s} ds = \int_T^\infty e^{-Z_s} ds + \int_0^T e^{-Z_s} ds \\ &= e^{-Z_T} \int_0^\infty e^{-(Z_s+T-Z_T)} ds + \int_0^T e^{-Z_s} ds. \end{aligned}$$

Therefore,

$$A_\infty \stackrel{d}{=} MA'_\infty + Q, \quad (26)$$

where A'_∞ is a copy of A_∞ which is independent of (M, Q) . The latter means that A_∞ is a perpetuity (see [2] for the definition and recent results) generated by the random vector (M, Q) . To verify (26) note that $\{Z_{s+t} - Z_t : s \geq 0\}$ is a subordinator which is independent of $\{Z_v : v \leq t\}$ and has the same law as $\{Z_u : u \geq 0\}$. Hence, for any Borel sets $\mathcal{A} \in \mathbb{R}^2$ and $\mathcal{B} \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{P}\left\{(e^{-Z_T}, A_T) \in \mathcal{A}, \int_0^\infty e^{-(Z_s+T-Z_T)} ds \in \mathcal{B}\right\} \\ &= \int_0^\infty e^{-t} \mathbb{P}\left\{(e^{-Z_t}, A_t) \in \mathcal{A}, \int_0^\infty e^{-(Z_{s+t}-Z_t)} ds \in \mathcal{B}\right\} dt \\ &= \int_0^\infty e^{-t} \mathbb{P}\{(e^{-Z_t}, A_t) \in \mathcal{A}\} dt \mathbb{P}\{A'_\infty \in \mathcal{B}\} \\ &= \mathbb{P}\{(e^{-Z_T}, A_T) \in \mathcal{A}\} \mathbb{P}\{A'_\infty \in \mathcal{B}\}, \end{aligned}$$

and (26) follows.

Our next result generalizes Proposition 3.1 in [7] dealing with moments of Q , and a number of results concerning moments of $\int_0^\infty e^{-Z_t} dt = Q + A$ (see, for example, Proposition 3.3 in [30]).

Proposition 3.8. *For $\lambda > 0$ and $\mu \geq 0$*

$$\mathbb{E}Q^\lambda M^\mu = \frac{\lambda}{1 + \varphi(\lambda + \mu)} \mathbb{E}Q^{\lambda-1} M^\mu,$$

where $\varphi(s) := -\log \mathbb{E}e^{-sZ_1}$, $s \geq 0$. In particular,

$$a_{n,m} := \mathbb{E}Q^n M^m = \frac{n!}{\prod_{k=0}^n (1 + \varphi(m+k))}, \quad m, n \in \mathbb{N}_0, \quad (27)$$

$$b_{n,m} := \mathbb{E}Q^n A^m = \frac{n!m!}{\prod_{k=0}^n (1 + \varphi(m+k))\varphi(1) \cdots \varphi(m)}, \quad m, n \in \mathbb{N}_0.$$

The moment sequences $\{a_{m,n} : m, n \in \mathbb{N}_0\}$ and $\{b_{m,n} : m, n \in \mathbb{N}_0\}$ uniquely determine the laws of the random vectors (M, Q) and (A, Q) respectively.

Proof. For $t > 0$ define $A_t := \int_0^t e^{-Z_v} dv$. The following is essentially Eq. (3.1) in [7].

$$A_t^\lambda e^{-\mu Z_t} = \lambda \int_0^t (A_t - A_v)^{\lambda-1} e^{-\mu(Z_t - Z_v)} e^{-(\mu+1)Z_v} dv.$$

Since

$$(A_t - A_v)^{\lambda-1} e^{-\mu(Z_t - Z_v)} = e^{-(\lambda-1)Z_v} \left(\int_0^{t-v} e^{-(Z_{s+v} - Z_v)} ds \right)^{\lambda-1} e^{-\mu(Z_t - Z_v)},$$

and $\{Z_{s+v} - Z_v : s \geq 0\}$ is a subordinator which is independent of $\{Z_v : v \leq t\}$ and has the same law as $\{Z_t : t \geq 0\}$, we conclude that $(\int_0^{t-v} e^{-(Z_{s+v} - Z_v)} ds)^{\lambda-1} e^{-\mu(Z_t - Z_v)}$ has the same law as $A_{t-v}^{\lambda-1} e^{-\mu Z_{t-v}}$ and is independent of $e^{-(\lambda-1)Z_v}$. Therefore, using Fubini's theorem,

$$\begin{aligned} \mathbb{E}A_T^\lambda e^{-\mu Z_T} &= \int_0^\infty e^{-t} \mathbb{E}A_t^\lambda e^{-\mu Z_t} dt \\ &= \lambda \int_0^\infty e^{-t} \left(\int_0^t e^{-v\varphi(\lambda+\mu)} \mathbb{E}A_{t-v}^{\lambda-1} e^{-\mu Z_{t-v}} dv \right) dt \\ &= \lambda \int_0^\infty e^{-v\varphi(\lambda+\mu)} \left(\int_v^\infty e^{-t} \mathbb{E}A_{t-v}^{\lambda-1} e^{-\mu Z_{t-v}} dt \right) dv \\ &= \lambda \int_0^\infty e^{-v(\varphi(\lambda+\mu)+1)} dv \int_0^\infty e^{-u} \mathbb{E}A_u^{\lambda-1} e^{-\mu Z_u} du \\ &= \frac{\lambda}{1 + \varphi(\lambda + \mu)} \mathbb{E}A_T^{\lambda-1} e^{-\mu Z_T}. \end{aligned}$$

Starting with

$$\mathbb{E}e^{-\mu Z_T} = \int_0^\infty e^{-t} \mathbb{E}e^{-\mu Z_t} dt = \int_0^\infty e^{-t(1+\varphi(\mu))} dt = \frac{1}{1+\varphi(\mu)}, \quad (28)$$

the formula for $a_{n,m}$ follows by induction. To prove that the law of (M, Q) is uniquely determined by $\{a_{n,m} : n, m \in \mathbb{N}_0\}$, it suffices to check that the marginal laws are uniquely determined by the corresponding moment sequences (see Theorem 3 in [24]). Since $M \in [0, 1]$ almost surely, the law of M is trivially moment determinate. From (27) it follows that

$$\mathbb{E}Q^n = \frac{n!}{(1+\varphi(1)) \cdots (1+\varphi(n))}, \quad n \in \mathbb{N}.$$

Set $f_n := \mathbb{E}Q^n/n!$. The limit $f := \lim_{n \rightarrow \infty} f_n/f_{n+1}$ exists and is positive (it is finite, if Z_t is compound Poisson, otherwise it is infinite). By the Cauchy-Hadamard formula, $f = \sup\{r > 0 : \mathbb{E}e^{rQ} < \infty\}$. Therefore, the law of Q has finite exponential moments of some orders from which we deduce that this law is moment determinate.

According to Proposition 3.3 in [30], $\mathbb{E}A_\infty^m = m!/(\varphi(1) \cdots \varphi(m))$, $m \in \mathbb{N}_0$. In view of (26),

$$\begin{aligned} \mathbb{E}Q^n A^m &= \mathbb{E}Q^n M^m \mathbb{E}A_\infty^m \\ &= \frac{n! m!}{\prod_{k=0}^n (1+\varphi(m+k)) \varphi(1) \cdots \varphi(m)}, \quad m, n \in \mathbb{N}_0. \end{aligned}$$

In the same way as above for (M, Q) it can be checked that the law of (A, Q) is determined by the moment sequence. We omit the details. \square

3.3 A bivariate result

Assume that (7) holds, or, equivalently, that

$$w(n) := \frac{1}{\mathbb{P}\{\xi \geq n\}} = \left(\sum_{k=n}^\infty p_k \right)^{-1} \sim \frac{n^\alpha}{L(n)} \quad (29)$$

for some $\alpha \in (0, 1)$. Let T be an exponentially distributed random variable with mean 1, which is independent of a subordinator $\{U_t : t \geq 0\}$ with zero drift and Lévy measure (9).

It is well known and follows, for example, from our Proposition 3.1 that $N_n/w(n)$ weakly converges to the Mittag-Leffler distribution with parameter α . From (27) or from Proposition 3.1 in [7] we have

$$\mathbb{E} \left(\int_0^T e^{-U_t} dt \right)^n = \frac{n!}{\Gamma^n(1-\alpha) \Gamma(1+n\alpha)}, \quad n \in \mathbb{N}_0,$$

which means that $\int_0^T e^{-U_t} dt$ has Mittag-Leffler distribution with parameter α . Thus,

$$\frac{N_n}{w(n)} \xrightarrow{d} \int_0^T e^{-U_t} dt. \quad (30)$$

Let η_α be a beta-distributed random variable with parameters $1 - \alpha$ and α , i.e. with density $x \mapsto \pi^{-1} \sin(\pi\alpha) x^{-\alpha} (1-x)^{\alpha-1}$, $x \in (0, 1)$. It is well known (see, for example, Theorem 8.6.3 in [6]) that $(1 - S_{N_n-1}/n)^\alpha \xrightarrow{d} \eta_\alpha^\alpha$. It can be checked that

$$\mathbb{E}\eta_\alpha^{n\alpha} = \frac{\Gamma(\alpha(n-1)+1)}{\Gamma(1-\alpha)\Gamma(\alpha n+1)}, \quad n \in \mathbb{N}_0.$$

From (28) it follows that e^{-U_T} has the same moment sequence. Therefore, since the distribution of e^{-U_T} is concentrated on $[0, 1]$, it coincides with the distribution of η_α^α . Thus,

$$\left(1 - \frac{S_{N_n-1}}{n}\right)^\alpha \xrightarrow{d} e^{-U_T}. \quad (31)$$

Now we point out a bivariate result generalizing (30) and (31).

Proposition 3.9. *Suppose (7) holds. Then,*

$$w^{-1}(n)(w(n - S_{N_n-1}), N_n) \xrightarrow{d} (e^{-U_T}, \int_0^T e^{-U_t} dt),$$

where $\{U_t : t \geq 0\}$ is a subordinator with zero drift and Lévy measure (9).

Remark 3.10. Corollary 3.3 in [26] states that

$$\left(\frac{L(n)}{n^\alpha}(N_{n+1} - 1), 1 - \frac{S_{N_{n+1}-1}}{n}\right) \xrightarrow{d} (X, Y), \quad (32)$$

where the distribution of a random vector (X, Y) was defined by the moment sequence. Our proof of Proposition 3.9 is different from and simpler than Port's proof of (32).

Proof. According to Proposition 3.8 it suffices to verify that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}w^i(n - S_{N_n-1})N_n^j}{w^{i+j}(n)} = \frac{j! \Gamma(\alpha(i-1)+1)}{\Gamma^{j+1}(1-\alpha) \Gamma(\alpha(i+j)+1)}, \quad i, j \in \mathbb{N}_0. \quad (33)$$

For $i = 0$, Eq. (33) follows from (22). For $i \in \mathbb{N}$, Eq. (33) is checked as follows.

$$\begin{aligned}
& \mathbb{E}w^i(n - S_{N_n-1})N_n^j \\
&= \sum_{k=1}^n \sum_{l=0}^{n-1} w^i(n-l)k^j \mathbb{P}\{N_n = k, S_{k-1} = l\} \\
&= w^i(n)\mathbb{P}\{\xi \geq n\} + \sum_{l=1}^{n-1} w^i(n-l)\mathbb{P}\{\xi \geq n-l\} \sum_{k=2}^{l+1} k^j \mathbb{P}\{S_{k-1} = l\} \\
&= w^i(n)\mathbb{P}\{\xi \geq n\} + \sum_{l=1}^{n-1} w^{i-1}(n-l) \sum_{k=2}^{l+1} k^j \mathbb{P}\{S_{k-1} = l\}.
\end{aligned}$$

As on p. 26 in [1], define the function $f(x) := 0$ on $[0, 1)$ and $f(x) := (k+1)^j$ on $[k, k+1)$ for $k \in \mathbb{N}$, and set $F(t) := \int_0^t f(x) dx$. Then,

$$\sum_{l=1}^{n-1} \sum_{k=2}^{l+1} k^j \mathbb{P}\{S_{k-1} = l\} = \sum_{k=1}^{n-1} (k+1)^j \mathbb{P}\{N_n > k\} = \mathbb{E}F(N_n).$$

By Karamata's theorem, $F(t) \sim (j+1)^{-1}t^{j+1}$. Since $\lim_{n \rightarrow \infty} N_n = \infty$ almost surely and $(N_n/w(n))^{j+1} \xrightarrow{d} \xi_\alpha^{j+1}$, where ξ_α is Mittag-Leffler distributed with parameter α , we have

$$\frac{F(N_n)}{w^{j+1}(n)} \xrightarrow{d} \frac{\xi_\alpha^{j+1}}{j+1}. \quad (34)$$

By (22), $\lim_{n \rightarrow \infty} \mathbb{E}(N_n/w(n))^{j+2} = \mathbb{E}\xi_\alpha^{j+2} < \infty$. Therefore, the sequence $\{F(N_n)/w^{j+1}(n) : n \in \mathbb{N}\}$ is uniformly integrable which together with (34) implies

$$\mathbb{E}F(N_n) \sim \mathbb{E} \frac{\xi_\alpha^{j+1}}{j+1} w^{j+1}(n) \sim \frac{j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+(j+1)\alpha)} \frac{n^{\alpha(j+1)}}{L^{j+1}(n)}. \quad (35)$$

Thus, if $i = 1$, we have

$$\mathbb{E}w(n - S_{N_n-1})N_n^j \sim \frac{j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+(j+1)\alpha)} \frac{n^{\alpha(j+1)}}{L^{j+1}(n)},$$

and (33) follows. Assume now that $i \geq 2$. Since $w^{i-1}(n) \sim n^{\alpha(i-1)}/L^{i-1}(n)$, Corollary 1.7.3 in [6] yields

$$W(s) := \sum_{n=1}^{\infty} s^n w^{i-1}(n) \sim \frac{\Gamma(1+\alpha(i-1))}{(1-s)^{1+\alpha(i-1)}L^{i-1}((1-s)^{-1})}, \quad s \uparrow 1.$$

By the same Corollary, (35) implies

$$\begin{aligned} R(s) &:= \sum_{n=1}^{\infty} s^n \left(\sum_{k=2}^{n+1} k^j \mathbb{P}\{S_{k-1} = l\} \right) \\ &\sim \frac{j!}{\Gamma^{j+1}(1-\alpha)} \frac{1}{(1-s)^{\alpha(j+1)} L^{j+1} ((1-s)^{-1})}, \quad s \uparrow 1. \end{aligned}$$

Therefore,

$$W(s)R(s) \sim \frac{\Gamma(1+\alpha(i-1))j!}{\Gamma^{j+1}(1-\alpha)} \frac{1}{(1-s)^{1+\alpha(i+j)} L^{i+j} ((1-s)^{-1})}, \quad s \uparrow 1.$$

The sequence $\{w^{i-1}(n) : n \in \mathbb{N}\}$ is non-decreasing. Hence, the sequence $\{\sum_{l=1}^{n-1} w^{i-1}(n-l) \sum_{k=2}^{l+1} k^j \mathbb{P}\{S_{k-1} = l\} : n = 2, 3, \dots\}$ is non-decreasing too. Another appeal to Corollary 1.7.3 in [6] gives, as $n \rightarrow \infty$,

$$\sum_{l=1}^{n-1} w^{i-1}(n-l) \sum_{k=2}^{l+1} k^j \mathbb{P}\{S_{k-1} = l\} \sim \frac{\Gamma(1+\alpha(i-1))j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+\alpha(i+j))} \frac{n^{\alpha(i+j)}}{L^{i+j}(n)}.$$

From this, (33) follows. \square

4 Proof of Theorem 1.2

Our proof essentially relies upon the following classical result

$$\lim_{n \rightarrow \infty} \mathbb{P}\{n - S_{N_n-1} \leq k\} = m^{-1} \sum_{i=1}^k \mathbb{P}\{\xi \geq i\} =: \mathbb{P}\{W \leq k\}, \quad k \in \mathbb{N}. \quad (36)$$

In order to see why (36) holds, note that

$$\begin{aligned} \mathbb{P}\{n - S_{N_n-1} = k\} &= \sum_{i=1}^n \mathbb{P}\{S_{i-1} = n - k, S_i \geq n\} \\ &= \mathbb{P}\{\xi \geq k\} \sum_{i=0}^{n-k} \mathbb{P}\{S_i = n - k\} \\ &\rightarrow m^{-1} \mathbb{P}\{\xi \geq k\}, \quad n \rightarrow \infty, \end{aligned}$$

by the elementary renewal theorem, and (36) follows.

From (12) we conclude that

$$M_n - N_n \xrightarrow{d} M'_W - 1, \quad (37)$$

where W is a random variable with distribution (36) which is independent of $\{M'_n : n \in \mathbb{N}\}$. Therefore, for any sequence $\{d_n : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} d_n = \infty$,

$$\frac{M_n - N_n}{d_n} \xrightarrow{P} 0. \quad (38)$$

Assume that the distribution of ξ does not belong to the domain of attraction of any stable law with index $\alpha \in [1, 2]$. Then, as is well known, it is not possible to find sequences $x_n > 0$ and $y_n \in \mathbb{R}$ such that $(S_n - y_n)/x_n$ converges to a proper and non-degenerate law. In view of

$$\mathbb{P}\{N_n > m\} = \mathbb{P}\{S_m \leq n - 1\}, \quad (39)$$

the same is true for N_n (see Theorem 7 in [13] and/or Theorem 2 in [19] for more details), and according to (38), for M_n .

Assume that conditions (ii) of Theorem 1.2 hold. If $\sigma^2 = \infty$ and (6) holds with $\alpha = 2$, then arguing as in the proof of Theorem 2 in [19] we conclude that, with a_n and b_n defined in our Theorem 1.2,

$$\frac{N_n - b_n}{a_n} \Rightarrow \mu_2.$$

Theorem 5 in [13] (if $\sigma^2 < \infty$) and Theorem 7 in [13] (if (6) holds for some $\alpha \in [1, 2)$) leads to the same limiting relation (with corresponding a_n and b_n , and with μ_2 replaced by μ_α in the latter case).

In view of (38) the same limiting relations hold for M_n and, hence, by Lemma 2.1, for X_n . The proof of Theorem 1.2 is complete.

5 A probabilistic proof of Theorem 1.4

Set $Y_n := n - S_{N_n-1}$. The sequence of distributions of $\{M_n/\mathbb{E}M_n : n \in \mathbb{N}\}$ is tight. According to (42), $\mathbb{E}M_n \sim \text{const } w(n)$, where $w(n)$ is the same as in (29). Therefore, there exists a sequence $\{n_k : k \in \mathbb{N}\}$ such that $\lim_{k \rightarrow \infty} n_k = \infty$ and, as $k \rightarrow \infty$, $M_{n_k}/w(n_k)$ converges in law to a random variable Z , say, with a proper law. From $Y_n \xrightarrow{P} +\infty$ and the result of Lemma 2.2 we conclude that, as $k \rightarrow \infty$, $\widehat{M}_{Y_{n_k}}/w(n_k)$ converges in law to a random variable $Z'' \stackrel{d}{=} Z$. By Proposition 3.9, as $k \rightarrow \infty$,

$$\left(\frac{w(Y_{n_k})}{w(n_k)}, \frac{N_{n_k} - 1}{w(n_k)} \right) \xrightarrow{d} (M, Q) := (e^{-U_T}, \int_0^T e^{-U_t} dt).$$

Rewriting (12) in the form

$$\frac{M_{n_k}}{w(n_k)} = \frac{\widehat{M}_{Y_{n_k}}}{w(Y_{n_k})} \frac{w(Y_{n_k})}{w(n_k)} + \frac{N_{n_k} - 1}{w(n_k)}$$

we conclude that, as $k \rightarrow \infty$,

$$\left(\frac{\widehat{M}_{Y_{n_k}}}{w(Y_{n_k})}, \frac{w(Y_{n_k})}{w(n_k)}, \frac{N_{n_k} - 1}{w(n_k)} \right) \xrightarrow{d} (Z', M, Q),$$

where $Z' \stackrel{d}{=} Z$ and using characteristic functions it can be checked that Z' is independent of (M, Q) . Furthermore,

$$Z \stackrel{d}{=} MZ' + Q. \quad (40)$$

From (26) it follows that the distribution of $\int_0^\infty e^{-U_t} dt$ is a solution of (40). By Theorem 1.5 (i) in [31] this solution is unique. Therefore, we have proved that, as $k \rightarrow \infty$,

$$\frac{M_{n_k}}{w(n_k)} \xrightarrow{d} \int_0^\infty e^{-U_t} dt.$$

The same argument can be repeated for any sequence like n_k , and the proof is complete.

Combining the proof above with the results of Subsection 3.2 immediately give the following corollary.

Corollary 5.1. *Suppose (7) holds. Then,*

$$\left(\frac{M_n - N_n}{w(n - S_{N_n-1})}, \frac{w(n - S_{N_n-1})}{w(n)}, \frac{N_n}{w(n)} \right) \xrightarrow{d} \left(\int_0^\infty e^{-(U_{t+T} - U_T)} dt, e^{-U_T}, \int_0^T e^{-U_t} dt \right).$$

Furthermore, $(M_n - N_n)/w(n - S_{N_n-1})$ and $(w(n - S_{N_n-1})/w(n), N_n/w(n))$ are asymptotically independent, and

$$w_n^{-1}(M_n - N_n, N_n) \xrightarrow{d} \left(\int_T^\infty e^{-U_t} dt, \int_0^T e^{-U_t} dt \right).$$

6 An analytic proof of Theorem 1.4

Nothing more than (1) and (2) is required for the proof given below. In particular, the construction in Section 2 is not needed.

For $k, n \in \mathbb{N}$ set $a_k(n) := \mathbb{E}X_n^k$ and $b_k(n) := \mathbb{E}N_n^k$. For $x \geq 0$ define

$$\Phi(x) := \frac{\Gamma(1-\alpha)\Gamma(\alpha x+1)}{\Gamma(\alpha(x-1)+1)} - 1 = \alpha x B(\alpha x, 1-\alpha) - 1,$$

where B denotes the beta function. Note that

$$B(\alpha x, 1-\alpha) = \int_0^1 y^{\alpha x-1} (1-y)^{-\alpha} dy = \alpha^{-1} \int_0^\infty e^{-xy} (1-e^{-y/\alpha})^{-\alpha} dy$$

and, hence,

$$\begin{aligned} \Phi(x) &= \int_0^\infty x e^{-xy} (1-e^{-y/\alpha})^{-\alpha} dy - 1 \\ &= \int_0^\infty (1-e^{-y/\alpha})^{-\alpha} d(1-e^{-xy}) - 1 \\ &= \int_0^\infty (1-e^{-xy}) \frac{e^{-y/\alpha}}{(1-e^{-y/\alpha})^{\alpha+1}} dy. \end{aligned} \quad (41)$$

Thus, the function Φ is the Laplace exponent of an infinitely divisible law with zero drift and Lévy measure ν given in (9). Note that (41) corrects an error on p. 102 in [4]. Assuming that (7) holds we will prove that

$$\lim_{n \rightarrow \infty} \frac{L^k(n)}{n^{\alpha k}} a_k(n) = \frac{k!}{\Phi(1) \cdots \Phi(k)} =: a_k, \quad k \in \mathbb{N}. \quad (42)$$

This will imply (see, for example, [4]) that (i) $a_k = \mathbb{E}(\eta^k)$, $k \in \mathbb{N}$, where η is a random variable with distribution of the exponential integral of a subordinator with zero drift and Lévy measure ν , and that (ii) the moments a_1, a_2, \dots uniquely determine the law of η . Note that the statement in (i) was first obtained in Example 3.4 in [30]. From (i) and (ii) it will follow that (42) implies (8).

Exactly in the same way as for $b_k(n)$ in the proof of Lemma 3.4, but starting with (1) instead of (16), it follows that

$$a_1(n) = 1 + r_n \sum_{i=1}^{n-1} a_1(n-i) p_i,$$

and, for $k \in \{2, 3, \dots\}$,

$$\begin{aligned} a_k(n) &= D_k(a_1(n), \dots, a_{k-2}(n)) + ka_{k-1}(n) + r_n \sum_{i=1}^{n-1} a_k(n-i)p_i \\ &=: d_k(n) + r_n \sum_{i=1}^{n-1} a_k(n-i)p_i, \end{aligned} \quad (43)$$

where the $D_k(\cdot)$ are the same as in the proof of Lemma 3.4, and $r_n := 1/(p_1 + \dots + p_{n-1})$. We are ready to prove (42). Again, we use induction on k . Suppose (42) holds for $k \in \{1, 2, \dots, m-1\}$. Set

$$\beta_1 := \frac{1}{1-b_1} \quad \text{and} \quad \beta_k := \frac{1}{b_{k-1} - k^{-1}b_k} \prod_{i=1}^{k-1} \frac{b_{i-1}}{b_{i-1} - i^{-1}b_i}, \quad k \in \{2, 3, \dots\},$$

where $b_k := k!/(\Gamma^k(1-\alpha)\Gamma(1+\alpha k))$, $k \in \mathbb{N}$, and note that

$$a_{k-1} - \beta_k(b_{k-1} - k^{-1}b_k) = 0. \quad (44)$$

In the following we exploit an idea given in the proof of Proposition 3 in [14]. Suppose there exists an $\epsilon > 0$ such that $a_k(n) > (\beta_k + \epsilon)b_k(n)$ for infinitely many n . It is possible to decrease ϵ so that the inequality $a_k(n) > (\beta_k + \epsilon)b_k(n) + c$ holds infinitely often for any fixed positive c . Thus, we can define $n_c := \inf\{n \geq 1 : a_k(n) > (\beta_k + \epsilon)b_k(n) + c\}$. Then

$$a_k(n) \leq (\beta_k + \epsilon)b_k(n) + c \quad \text{for all } n \in \{1, 2, \dots, n_c - 1\}. \quad (45)$$

We have

$$\begin{aligned} (\beta_k + \epsilon)b_k(n_c) + c &< a_k(n_c) \stackrel{(43)}{=} d_k(n_c) + r_{n_c} \sum_{i=1}^{n_c-1} a_k(n_c-i)p_i \\ &\stackrel{(45)}{\leq} d_k(n_c) + c + (\beta_k + \epsilon)r_{n_c} \sum_{i=1}^{n_c-1} b_k(n_c-i)p_i \\ &\stackrel{(43), (15)}{=} D_k(a) + ka_{k-1}(n_c) + c \\ &\quad + (\beta_k + \epsilon)(r_{n_c} - 1)(b_k(n_c) - D_k(b) - kb_{k-1}(n_c)) + \\ &\quad + (\beta_k + \epsilon)b_k(n_c) - (\beta_k + \epsilon)(D_k(b) + kb_{k-1}(n_c)), \end{aligned}$$

or, equivalently,

$$\begin{aligned} 0 &< D_k(a) + ka_{k-1}(n_c) + (\beta_k + \epsilon)(r_{n_c} - 1)(b_k(n_c) - D_k(b) - kb_{k-1}(n_c)) \\ &\quad - (\beta_k + \epsilon)(D_k(b) + kb_{k-1}(n_c)), \end{aligned}$$

where we have used the abbreviations $D_k(a) := D_k(a_1(n_c), \dots, a_{k-2}(n_c))$ and $D_k(b) := D_k(b_1(n_c), \dots, b_{k-2}(n_c))$ for convenience. Divide the latter inequality by $z(c) := n_c^{(k-1)\alpha}/L^{k-1}(n_c)$ and let c go to ∞ (which implies $n_c \rightarrow \infty$). Notice that, according to (7), $r_n - 1 \sim n^{-\alpha}L(n)$ and that by the induction assumption

$$\lim_{c \rightarrow \infty} \frac{D_k(a_1(n_c), \dots, a_{k-2}(n_c))}{z(c)} = 0 \quad \text{and} \quad \lim_{c \rightarrow \infty} \frac{a_{k-1}(n_c)}{z(c)} = a_{k-1}.$$

Using these facts and (22) we obtain

$$0 \leq ka_{k-1} + (\beta_k + \epsilon)b_k - (\beta_k + \epsilon)kb_{k-1}.$$

Since the function Φ defined at the beginning of the proof is positive for $x > 0$, and $kb_{k-1}/b_k - 1 = \Phi(k)$, we conclude that $kb_{k-1} - b_k > 0$. Therefore,

$$\epsilon(kb_{k-1} - b_k) \leq k(a_{k-1} - \beta_k(b_{k-1} - k^{-1}b_k)) = 0$$

by (44). This is the desired contradiction. Thus, we have verified that

$$\limsup_{n \rightarrow \infty} \frac{a_k(n)}{b_k(n)} \leq \beta_k.$$

A symmetric argument proves the converse inequality for the lower bound. Therefore,

$$a_k(n) \sim \beta_k b_k(n) \sim \beta_k b_k \frac{n^{k\alpha}}{L^k(n)} = a_k \frac{n^{k\alpha}}{L^k(n)}.$$

A similar but simpler reasoning yields the result for $k = 1$. We omit the details. The proof is complete.

7 Proof of Theorem 1.1

By Lemma 2.1 it suffices to prove the result for M_n . Assume first that $m < \infty$. It is well known that

$$\lim_{n \rightarrow \infty} \frac{N_n}{n} = \frac{1}{m} \quad \text{almost surely.} \quad (46)$$

In view of (37), $\lim_{n \rightarrow \infty} (M_n - N_n)/n = 0$ almost surely, which yields $\lim_{n \rightarrow \infty} M_n/n = 1/m$ almost surely. By the elementary renewal theorem, $\mathbb{E}N_n \sim n/m$. Using the same approach as in Section 6 it is straightforward

to check that $\mathbb{E}M_n \sim n/m$. Conversely, if $M_n/a_n \xrightarrow{P} 1$, then (38) gives $(M_n - N_n)/a_n \xrightarrow{P} 0$. Therefore, $N_n/a_n \xrightarrow{P} 1$. An appeal to (46) allows us to conclude that $a_n \sim n/m$.

Assume now that $m = \infty$. According to (23), $\mathbb{E}N_n^k \sim n^k/L^k(n)$, $k \in \mathbb{N}$. Again, the same approach as in Section 6 yields

$$\mathbb{E}X_n^k \sim \frac{n^k}{L^k(n)} \sim (\mathbb{E}X_n)^k, \quad k \in \mathbb{N}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\frac{X_n}{\mathbb{E}X_n} \right)^k = 1, \quad k \in \mathbb{N},$$

which proves (4). In fact, to arrive at (4), it suffices to know that $\mathbb{E}X_n \sim n/L(n)$ and $\mathbb{E}X_n^2 \sim n^2/L^2(n)$ and exploit Chebyshev's inequality. The proof is complete.

8 Proof of Theorem 1.5

By Theorem 3 (c) and formulae on p. 42 in [5] (see also [18])

$$\frac{N_n - b(n) - 1}{a(n)} \Rightarrow \mu_1,$$

where μ_1 is the 1-stable law with characteristic function $\int_{-\infty}^{\infty} e^{itx} \mu_1(dx) = \exp(it \log |t| - |t|\pi/2)$, $t \in \mathbb{R}$. By Corollary 3.6,

$$\frac{M_n}{N_n - 1} \xrightarrow{P} 1. \tag{47}$$

Therefore,

$$\frac{M_n - b(n)}{a(n)} - \frac{M_n - N_n + 1}{N_n - 1} \frac{b(n)}{a(n)} \Rightarrow \mu_1.$$

Thus, to prove the theorem it suffices to show that the second summand tends to 0 in probability. Clearly, this can be regarded as a rate of convergence result for (47). Recalling the notation $Y_n = n - S_{N_n-1}$ and using (12) gives

$$\begin{aligned} \frac{M_n - N_n + 1}{N_n - 1} \frac{b(n)}{a(n)} &= \frac{\widehat{M}_{Y_n}}{Y_n/m(Y_n)} \frac{m(n)}{m(Y_n)} \frac{b(n)Y_n}{na(n)} \frac{n}{m(n)(N_n - 1)} \\ &=: \prod_{i=1}^4 K_i(n). \end{aligned}$$

By Corollary 3.6, $m(n)M_n/n \xrightarrow{P} 1$. Using the equality of distributions (12) and the fact that $Y_n \xrightarrow{P} \infty$ allows us to conclude that $K_1(n) \xrightarrow{P} 1$. By Theorem 6 in [12], $K_2(n) \xrightarrow{d} 1/R$, where R is a random variable uniformly distributed on $[0, 1]$. By Proposition 3.7, $K_3(n) \xrightarrow{P} 0$. Finally, by Corollary 3.6, $K_4(n) \xrightarrow{P} 1$. The proof is complete.

9 Number of collisions in beta coalescents

In this section the main results presented in Section 1 are applied to the number of collisions that take place in beta coalescent processes until there is just a single block. Other closely related functionals of coalescent processes such as the total branch length or the number of segregating sites have been studied in [10] and [21].

Let \mathcal{E} denote the set of all equivalence relations on \mathbb{N} . For $n \in \mathbb{N}$ let $\varrho_n : \mathcal{E} \rightarrow \mathcal{E}_n$ denote the natural restriction to the set \mathcal{E}_n of all equivalence relations on $\{1, \dots, n\}$. For $\eta \in \mathcal{E}_n$ let $|\eta|$ denote the number of blocks (equivalence classes) of η .

Pitman [25] and Sagitov [29] independently introduced coalescent processes with multiple collisions. These Markovian processes with state space \mathcal{E} are characterized by a finite measure Λ on $[0, 1]$ and are, hence, also called Λ -coalescent processes. For a Λ -coalescent $\{\Pi_t : t \geq 0\}$, it is known that the process $\{|\varrho_n \Pi_t| : t \geq 0\}$ has infinitesimal rates

$$g_{nk} := \lim_{t \downarrow 0} \frac{\mathbb{P}\{|\varrho_n \Pi_t| = k\}}{t} = \binom{n}{k-1} \int_{[0,1]} x^{n-k-1} (1-x)^{k-1} \Lambda(dx) \quad (48)$$

for all $k, n \in \mathbb{N}$ with $k < n$. Let $g_n := \sum_{k=1}^{n-1} g_{nk}$, $n \in \mathbb{N}$, denote the total rates. We are interested in the number X_n of collisions (jumps) that take place in the restricted coalescent process $\{|\varrho_n \Pi_t| : t \geq 0\}$ until there is just a single block. From the structure of the coalescent process it follows that $(X_n)_{n \in \mathbb{N}}$ satisfies the recursion (1), where I_n is independent of X_2, \dots, X_{n-1} with distribution $\mathbb{P}\{I_n = k\} = g_{n,n-k}/g_n$, $k \in \{1, \dots, n-1\}$. The random variable $n - I_n$ is the (random) state of the process $\{|\varrho_n \Pi_t| : t \geq 0\}$ after its first jump.

We consider beta coalescents, where, by definition, $\Lambda = \beta(a, b)$ is the beta distribution with density $x \mapsto (B(a, b))^{-1} x^{a-1} (1-x)^{b-1}$ with respect to the Lebesgue measure on $(0, 1)$, and $B(a, b) := \Gamma(a)\Gamma(b)/\Gamma(a+b)$ denotes

the beta function, $a, b > 0$. In this case the rates (48) have the form

$$\begin{aligned} g_{nk} &= \binom{n}{k-1} \frac{1}{B(a, b)} \int_0^1 x^{a+n-k-2} (1-x)^{b+k-2} dx \\ &= \binom{n}{k-1} \frac{B(a+n-k-1, b+k-1)}{B(a, b)}, \quad k, n \in \mathbb{N}, k < n. \end{aligned} \quad (49)$$

From

$$g_{k+1, k} = \frac{k(k+1)}{2} \frac{B(a, b+k-1)}{B(a, b)}$$

it follows that

$$g_n = \sum_{k=1}^{n-1} (g_{k+1} - g_k) = \sum_{k=1}^{n-1} \frac{2}{k+1} g_{k+1, k} = \frac{1}{B(a, b)} \sum_{k=1}^{n-1} k B(a, b+k-1).$$

In the following it is assumed that $b = 1$ such that the rates (49) reduce to

$$g_{nk} = \binom{n}{k-1} \frac{B(a+n-k-1, k)}{B(a, 1)} = \frac{n!}{(n-k+1)!} a \frac{\Gamma(a+n-k-1)}{\Gamma(a+n-1)},$$

and the total rates to

$$g_n = a \sum_{k=1}^{n-1} k B(a, k) = \begin{cases} \frac{a}{a-2} \left(1 - \frac{\Gamma(a)\Gamma(n+1)}{\Gamma(a+n-1)} \right) & \text{for } a > 0, a \neq 2, \\ 2(h_n - 1) & \text{for } a = 2. \end{cases}$$

Here, $h_n := \sum_{i=1}^n 1/i$ denotes the n -th harmonic number. From the last formula it follows that the parameter $a = 2$ plays a special role in this model. Define

$$p_k := \frac{(2-a)\Gamma(a+k-1)}{\Gamma(a)\Gamma(k+2)}, \quad k \in \mathbb{N}. \quad (50)$$

Assume now that $0 < a < 2$. In this case (and only in this case) we have $p_k \geq 0$ for $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} p_k = 1$. Let ξ be a random variable with distribution $\mathbb{P}\{\xi = k\} = p_k$, $k \in \mathbb{N}$. For $0 < a < 2$, $a \neq 1$, we can rewrite (50) in terms of $\alpha := 2 - a$ in the form

$$p_k = \frac{1}{1-\alpha} \binom{\alpha}{k+1} (-1)^k, \quad k \in \mathbb{N}.$$

Therefore, for $a \neq 1$, i.e. $\alpha \neq 1$, ξ has probability generating function

$$\mathbb{E}s^\xi = \sum_{k=1}^{\infty} p_k s^k = \frac{1}{1-\alpha} \sum_{k=1}^{\infty} \binom{\alpha}{k+1} (-s)^k = \frac{1 - \alpha s - (1-s)^\alpha}{(1-\alpha)s}.$$

For $a = 1$, i.e. $\alpha = 1$, the probability generating function is

$$\mathbb{E}s^\xi = \sum_{k=1}^{\infty} \frac{s^k}{k(k+1)} = 1 - \log(1-s) + \frac{\log(1-s)}{s}$$

with continuous extensions for $s = 0$ and $s = 1$. For $0 < a < 2$ it follows by induction on n that

$$\mathbb{P}\{\xi \geq n\} = \frac{\Gamma(a+n-1)}{\Gamma(a)\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

Using $\Gamma(n+x) \sim \Gamma(n)n^x$ for $n \rightarrow \infty$, we conclude that

$$\mathbb{P}\{\xi \geq n\} \sim \frac{n^{a-2}}{\Gamma(a)} = \frac{n^{-\alpha}}{\Gamma(2-\alpha)}, \quad n \rightarrow \infty.$$

Thus, if $1 < a < 2$, or, equivalently, $0 < \alpha < 1$, Theorem 1.4 is applicable (with $L(n) \equiv 1/\Gamma(a) = 1/\Gamma(2-\alpha)$), and we obtain the following result.

Theorem 9.1. *For the $\beta(a,1)$ -coalescent with $1 < a < 2$, i.e., $0 < \alpha := 2-a < 1$, the number X_n of collision events satisfies*

$$\frac{X_n}{\Gamma(2-\alpha)n^\alpha} \xrightarrow{d} \int_0^\infty e^{-U_t} dt,$$

where $\{U_t : t \geq 0\}$ is a subordinator with zero drift and Lévy measure (9).

Note that, for $\Lambda = \beta(a,b)$, we have $\mu_{-1} := \int x^{-1} \Lambda(dx) < \infty$ if and only if $a > 1$. Under the condition $\mu_{-1} < \infty$, limiting results similar to that presented in the above Theorem 9.1 are known for the number of segregating sites (see, for example, Proposition 5.1 in [21]) for general Λ -coalescent processes with mutation.

Assume now that $0 < a < 1$. Then, $m := \mathbb{E}\xi = 1/(1-a) < \infty$. It is straightforward to verify that

$$\sum_{k=1}^n k^2 p_k \sim \frac{2-a}{\Gamma(a+1)} n^a, \quad n \rightarrow \infty.$$

In particular, the variance of ξ is infinite. Thus, Theorem 1.2 is applicable (with $L(n) \equiv (2-a)/\Gamma(a+1) = \alpha/\Gamma(3-\alpha)$, $C := 1/\Gamma(a) = 1/\Gamma(2-\alpha)$, $b_n := n(1-a) = n(\alpha-1)$ and $c_n := n^{1/\alpha}$), and yields the following result.

Theorem 9.2. *For the $\beta(a, 1)$ -coalescent with $0 < a < 1$, i.e., $1 < \alpha := 2 - a < 2$, the number X_n of collision events satisfies*

$$\frac{X_n - n(\alpha - 1)}{(\alpha - 1)^{(\alpha+1)/\alpha} n^{1/\alpha}} \Rightarrow \mu_\alpha,$$

or, equivalently,

$$\frac{X_n - n(\alpha - 1)}{(\alpha - 1)n^{1/\alpha}} \xrightarrow{d} S_\alpha, \quad (51)$$

where $\mathbb{E} \exp(itS_\alpha) = \exp(|t|^\alpha (\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sgn}(t)))$, $t \in \mathbb{R}$.

Gnedin and Yakubovich [17, Theorem 9] use analytic methods to verify the same convergence result (51) for Λ -coalescents satisfying $\Lambda([0, x]) = Ax^a + O(x^{a+\zeta})$, $x \rightarrow 0$, $0 < a < 1$, $\zeta > \max\{(2-a)^2/(5-5a+a^2), 1-a\}$.

Theorems 9.1 and 9.2 do not cover the asymptotics of X_n for the Boltzhausen-Sznitman coalescent, i.e. the $\beta(a, b)$ -coalescent with $a = b = 1$. The limiting behaviour of X_n for the Boltzhausen-Sznitman coalescent was studied in [20], and follows also from our Theorem 1.5 with $p_k := 1/(k(k+1))$, $L(n) \equiv 1$, $c(x) := x$, $b(x) := x/\log x + x \log \log x / (\log x)^2$, and $a(x) := b^2(x)/x \sim x/(\log x)^2$. Therefore, the asymptotics of X_n for all $\beta(a, 1)$ -coalescent processes with $0 < a < 2$ is clarified. Unfortunately, our method cannot be used to treat the asymptotics of X_n for $\beta(a, 1)$ -coalescent processes with $a \geq 2$, as in this case the crucial assumption (2) is not satisfied.

10 Possible generalizations

We have studied random recursions (1) under the assumption that

$$I_n \xrightarrow{d} \xi \quad (52)$$

with specified rate of convergence (2). If $\mathbb{E}\xi < \infty$, this specific rate of convergence (2) ensures that X_n and N_n have the same limiting behaviour. Under the sole condition (52) without any assumption on the speed of convergence such as (2), the asymptotics of X_n can differ significantly from that of N_n , even if $\mathbb{E}\xi < \infty$. Assume for example that $I_2 \equiv 1$ and that $\mathbb{P}\{I_n = n-1\} = 1 - \mathbb{P}\{I_n = 1\} = 1/n$ for $n \geq 3$, or, equivalently, that $\pi_{n,1} = 1 - \pi_{n,n-1} = 1/n$ for $n \geq 3$. In this case, (52) is obviously satisfied with $\xi \equiv 1$. In particular, $S_k \equiv k$, $k \in \mathbb{N}_0$, and $N_n \equiv n$, $n \in \mathbb{N}$. It is straightforward to derive the distribution of X_n . We have $\mathbb{P}\{X_n = n-1\} = \prod_{i=2}^n \pi_{i,i-1} = 2/n$ and, for $k \in \{1, \dots, n-2\}$, $\mathbb{P}\{X_n = k\} = \pi_{n-k+1,1} \prod_{i=n-k+2}^n \pi_{i,i-1} = 1/n$. Thus,

X_n/n is asymptotically uniformly distributed on $(0, 1)$. In particular, N_n and X_n do not have a similar limiting behaviour.

It is even more evident that the rate of convergence in (52) will influence the limiting behaviour of X_n , if $\mathbb{E}\xi = \infty$, in particular, when (6) holds.

For the case $\mathbb{E}\xi = \infty$ we left open the interesting theoretical problem of finding necessary and sufficient conditions under which $(X_n - b_n)/a_n$ weakly converges to a proper law. Theorems 1.1, 1.4 and 1.5 are our contribution to the one-sided solution of this problem. To solve the problem in full generality one should, among others, understand a weak behaviour of X_n under the assumption $\sum_{k=n}^{\infty} p_k \sim 1/L(n)$, where L is some slowly varying function. It seems that this case is not amenable to the analysis presented in this work.

We concentrated on M_n , the number of jumps of the process $R^{(n)} := \{R_k^{(n)} : k \in \mathbb{N}_0\}$, which is an interesting generalization of random walks. We think it is of interest to analyse other functionals of $R^{(n)}$ such as $M_n^{(i)} := \#\{k \geq 1 : R_k^{(n)} - R_{k-1}^{(n)} = i\}$ for some fixed $i \in \{0, 1, \dots, n-1\}$, or $T_n := M_n + M_n^{(0)} = \inf\{k \geq 1 : R_k^{(n)} = n-1\}$. P. Negadajlov has already checked that Theorems 1.1, 1.2, 1.4 and 1.5 of the present work remain valid with $X_n \stackrel{d}{=} M_n$ replaced with T_n .

Acknowledgement. The authors thank Alexander Gnedin for fruitful comments and discussions, in particular, for pointing out an error in Section 10 of the first version of the manuscript.

References

- [1] ALSMEYER, G. (1991). Some relations between harmonic renewal measures and certain first passage times. *Stat. Prob. Lett.* **12**, 19–27.
- [2] ALSMEYER, G., IKSANOV, A., AND ROESLER, U. (2007). On distributional properties of perpetuities. Submitted to *J. Theor. Prob.*
- [3] BARBOUR, A. D. AND GNEDIN, A. V. (2006). Regenerative compositions in the case of slow variation. *Stoch. Process. Appl.* **116**, 1012–1047.
- [4] BERTOIN, J. AND YOR, M. (2001). On subordinators, self-similar Markov processes and some factorization of the exponential variable. *Electron. Commun. Probab.* **6**, 95–106.
- [5] BINGHAM, N. H. (1972). Limit theorems for regenerative phenomena, recurrent events and renewal theory. *Z. Wahrsch. verw. Geb.* **21**, 20–44.
- [6] BINGHAM N. H., GOLDIE C. M., AND TEUGELS, J. L. (1989). *Regular variation*. Cambridge: Cambridge University Press.

- [7] CARMONA, P., PETIT, F., AND YOR, M. (1997). On the distribution and asymptotic results for exponential functionals of Lévy processes. In: M. Yor (editor) *Exponential functionals and principal values related to Brownian motion*, 73–121. Biblioteca de la Revista Matemática Iberoamericana.
- [8] VAN CUTSEM, B. AND YCART, B. (1994). Renewal-type behaviour of absorption times in Markov chains. *Adv. Appl. Prob.* **26**, 988–1005.
- [9] DARLING, D. A. AND KAC, M. (1957). On occupation-times for Markov processes. *Trans. Amer. Math. Soc.* **84**, 444–458.
- [10] DRMOTA, M., IKSANOV, A., MOEHLE, M., AND ROESLER, U. (2007). Asymptotic results concerning the total branch length of the Bolthausen-Sznitman coalescent. *Stoch. Process. Appl.* **117**, to appear.
- [11] DRMOTA, M., IKSANOV, A., MOEHLE, M., AND ROESLER, U. (2006). A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree. Submitted to *Random Struct. Algorithms*.
- [12] ERICKSON, K. B. (1970). Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151**, 263–291.
- [13] FELLER, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67**, 98–119.
- [14] GNEDIN, A. V. (2004). The Bernoulli sieve. *Bernoulli* **10**, 79–96.
- [15] GNEDIN, A., PITMAN, J., AND YOR, M. (2006). Asymptotic laws for regenerative compositions: gamma subordinators and the like. *Probab. Theory Relat. Fields* **135**, 576–602.
- [16] GNEDIN, A., PITMAN, J., AND YOR, M. (2006). Asymptotic laws for compositions derived from transformed subordinators. *Ann. Probab.* **34**, 468–492.
- [17] GNEDIN, A. AND YAKUBOVICH, Y. (2007). On the number of collisions in Λ -coalescents. Preprint
- [18] DE HAAN, L. AND RESNICK, S. I. (1979). Conjugate II-variation and process inversion. *Ann. Probab.* **7**, 1028–1035.
- [19] HEYDE, C. C. (1967). A limit theorem for random walks with drift. *J. Appl. Prob.* **4**, 144–150.
- [20] IKSANOV, A. AND MÖHLE, M. (2007). A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree. *Electron. Commun. Probab.* **12**, 28–35.
- [21] MÖHLE, M. (2006). On the number of segregating sites for populations with large family sizes. *Adv. Appl. Prob.* **38**, 750–767.
- [22] NEININGER, R. AND RÜSCHENDORF, L. (2004). On the contraction method with degenerate limit equation. *Ann. Probab.* **32**, 2838–2856.

- [23] PANHOLZER, A. (2006). Cutting down very simple trees. *Quest. Math.* **29**, 211–227.
- [24] PETERSEN, L. C. (1982). On the relation between the multidimensional moment problem and the one-dimensional moment problem. **51**, 361–366.
- [25] PITMAN, J. (1999). Coalescents with multiple collisions. *Ann. Probab.* **27**, 1870–1902.
- [26] PORT, S. C. (1964). Some theorems on functionals of Markov chains. *Ann. Math. Stat.* **35**, 1275–1290.
- [27] RÖSLER, U. (1991). A limit theorem for "Quicksort". *RAIRO, Inform. Theor. Appl.* **25**, 85–100.
- [28] RÖSLER, U. AND RÜSCHENDORF, L. (2001). The contraction method for recursive algorithms. *Algorithmica* **29**, 3–33.
- [29] SAGITOV, S. (1999). The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Prob.* **36**, 1116–1125.
- [30] URBANIK, K. (1992). Functionals on transient stochastic processes with independent increments. *Studia Math.* **103**, 299–315.
- [31] VERVAAT, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Prob.* **11**, 750–783.